($Nu = 1$) to convection ($Nu > 1$) were verified in the present study for L/D of 0·5 and 1·0. Transient data obtained during phase change are shown in Fig. 3 and these substantiate the effect of L/D in the range of 0·59–3·0.

Experimental interfacial velocities and positions are given in Fig. 4. Also shown are predicted results using k_{eff} as a function of L/D in the finite-difference equations. The agreement is superior to that obtained when the effect of L/D is ignored.

In summary, the additional uses of the numerical method of Murray and Landis now include (1) Cases in which only one phase is present initially; (2) Cases with unequal densities for the phases; (3) Cases with free convection in the liquid; and (4) Cases such that the depth-width ratio for the liquid affects convection in the liquid.

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REFERENCES

1. W. D. Murray and F. Landis, Numerical and machine solutions of transient heat conduction problems involving phase change, *J. Heat Transfer* **81**, 106–112 (1959).
2. L. J. Thomas and J. W. Westwater, Microscopic study of solid–liquid interfaces during melting and freezing, *Chem. Engng Prog. Symp. Ser.* **59**, No. 41, 155–164 (1963).
3. D. V. Boger and J. W. Westwater, Effect of buoyancy on the melting and freezing process, *J. Heat Transfer* **89**, 81–89 (1967).
4. W. L. Heitz, Hydrodynamic stability of water and its effect on melting and freezing, Ph.D. Thesis, University of Illinois, Urbana, Illinois (1970).
5. E. R. G. Eckert and R. M. Drake, Jr., *Heat and Mass Transfer*. McGraw-Hill, New York (1959).
6. P. A. Longwell, Graphical method for solution of freezing problems, *J. Am. Inst. Chem. Engrs* **4**, 53–57 (1958).
7. L. E. Scriven, On the dynamics of phase growth, *Chem. Engng Sci.* **10**, 1–13 (1959).
8. S. Globe and D. Dropkin, Natural-convection heat transfer in liquids confined by two horizontal plates and heated from below, *J. Heat Transfer* **81**, 24–28 (1959).
9. E. Schmidt and P. L. Silveston, Natural convection in horizontal liquid layers, *Chem. Engng Prog. Symp. Ser.* **55**, No. 29, 163–169 (1959).
10. J. L. O'Toole and P. L. Silveston, Correlations of convective heat transfer in confined horizontal layers, *Chem. Engng Prog. Symp. Ser.* **57**, No. 32, 81–86 (1961).
11. I. Catton and D. K. Edwards, Effect of side walls on natural convection between horizontal plates heated from below, *J. Heat Transfer* **89**, 295–299 (1967).

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STEADY STATE TEMPERATURE PROFILES WITHIN INSULATED ELECTRICAL CABLES HAVING VARIABLE CONDUCTIVITIES

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NOMENCLATURE

- E, electric field;
- C_0, constant defined by equation (6);
- f_1, defined by equation (18);
- f_2, defined by equation (19);
- f_3, solution of equation (16);
- g_0, defined by equation (39);
- g_1, defined by equation (40);
- g_2, solution of equation (36);
- h, $\gamma/\ln \eta_0$;
- J, Joule heating parameter, $\sigma_0 E^2 r_c^2/k_0 T_0$;
- k, thermal conductivity of conductor at temperature T;
- k_0, thermal conductivity of conductor at temperature T_0;
- k_i, thermal conductivity of insulator;
- r, radial coordinate measured from center of conductor;
- r_c, radius of conductor;
- r_0, radial distance from center of conductor to outside surface of insulator;
- T, temperature;
- T_0, temperature of outside surface of insulator.

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Greek symbols

α, product of temperature coefficient of thermal conductivity times T_0;
β, product of temperature coefficient of electrical conductivity times T_0;
γ, k_i/k;
ϵ, J^{-1};
θ, $(T - T_0)/T_0$ for conductor, $0 \leqslant \eta \leqslant 1$;
θ_i, $(T - T_0)/T_0$ for insulator $1 \leqslant \eta \leqslant \eta_0$;
η, r/r_c;
η_0, r_0/r_c;
$\tilde{\eta}$, $(\eta - 1)/\epsilon^{\frac{1}{2}}$;
σ, electrical conductivity of conductor at temperature T;
σ_0, electrical conductivity of conductor at temperature T_0.

INTRODUCTION

NONLINEAR effects associated with Joule heating are exploited in various analytical detection techniques [1, 2]. Often the electrical conductor is surrounded by a thin layer of gas, or an insulator, whose surface is kept at a fixed temperature. Furthermore it is desirable to predict the temperature profile within an electrical cable and its insulator, for systems within which high Joule heating may occur. Bird *et al.* [3] have reported L. F. J. Broer's analysis concerning low Joule heating within an electrical cable having variable conductivities, when the surface of the conductor is kept at a constant temperature. Therefore, it is of interest to extend Broer's analysis to include an insulator and also to treat the case of high Joule heating.

ANALYSIS

Consider an insulated d.c. electrical cable with temperature dependent electrical and thermal conductivities. The insulator has a constant thermal conductivity and has its outer surface maintained at a constant temperature, T_0. The steady state heat conduction equations for the cable and for the insulator are respectively,

$$\frac{1}{\eta}\frac{\mathrm{d}}{\mathrm{d}\eta}\left[\eta(1-\alpha\theta)\frac{\mathrm{d}\theta}{\mathrm{d}\eta}\right] + (1-\beta\theta)J = 0, \qquad 0 \leqslant \eta \leqslant 1, \tag{1}$$

and

$$\frac{1}{\eta}\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta\frac{\mathrm{d}\theta_i}{\mathrm{d}\eta}\right) = 0, \qquad 1 \leqslant \eta \leqslant \eta_0. \tag{2}$$

The quantities $(1 - \alpha\theta)$ and $(1 - \beta\theta)$ represent the ratio of the thermal conductivity to that at the reference temperature T_0 and the ratio of the electrical conductivity to that at the temperature T_0 respectively. For conductors which obey the Wiedemann–Franc–Lorentz Law, it is readily shown that $\beta - \alpha = 1$. The quantity J represents the rate of Joule heating per unit volume of conductor divided by the rate of heat lost per unit surface area of conductor.

Both the temperature and the thermal flux are considered to be finite and continuous functions of the radial coordinate, η. The outside of the insulator is kept at a fixed temperature. Thus, the boundary conditions are

$$\theta = \theta_i \quad \text{at} \quad \eta = 1, \tag{3}$$

$$(1 - \alpha\theta)\theta' = \gamma\theta_i' \quad \text{at} \quad \eta = 1, \tag{4}$$

$$\theta_i = 0 \quad \text{at} \quad \eta = \eta_0. \tag{5}$$

Equations (2) and (5) imply that

$$\theta_i(\eta) = C_0 \ln(\eta/\eta_0), \qquad 1 \leqslant \eta \leqslant \eta_0. \tag{6}$$

Thus from equations (3) and (4) we have

$$\theta(1) = -C_0 \ln \eta_0, \tag{7}$$

$$[1 - \alpha\theta(1)]\theta'(1) = \gamma C_0, \tag{8}$$

where C_0 is a constant to be determined from the equations governing the temperature $\theta(\eta)$ in the conductor. Consequently, the problem is to find a continuous function $\theta(\eta)$ and a constant C_0 which satisfy equation (1) and the conditions (7) and (8), which together imply that

$$\theta'(1) = \frac{\gamma C_0}{1 + \alpha C_0 \ln \eta_0} \tag{9}$$

$$\theta'(1) + h\theta(1) = \alpha\theta(1)\theta'(1), \tag{10}$$

where $h = \gamma/\ln \eta_0$.

This problem is highly nonlinear and exact analytical techniques are not immediately applicable. Therefore asymptotic methods are employed to study two limiting cases: (i) low Joule heating where $0 < J \ll 1$, and (ii) high Joule heating where $J \gg 1$. In the case of low Joule heating, a straight-forward regular perturbation series is sufficient. However, the case of high Joule heating leads to a singular perturbation problem requiring boundary layer techniques.

(i) *Low Joule heating*

When $0 < J \ll 1$ we assume that

$$\theta = Jf_1(\eta) + J^2 f_2(\eta) + J^3 f_3(\eta) + \dots \tag{11}$$

Substituting equation (11) into equations (1) and (10) and equating like powers of J yields the following set of equations for the $f_i (i = 1, 2, 3, \dots)$:

$$\frac{1}{\eta}\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta\frac{\mathrm{d}f_1}{\mathrm{d}\eta}\right) = -1, \tag{12}$$

$$f_1'(1) + hf_1(1) = 0, \tag{13}$$

$$\frac{1}{\eta}\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta\frac{\mathrm{d}f_2}{\mathrm{d}\eta}\right) = (\beta - \alpha)f_1 + \alpha f_1'^2, \tag{14}$$

$$f_2'(1) + hf_2(1) = \alpha f_1(1)f_1'(1) \tag{15}$$

$$\frac{1}{\eta}\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta\frac{\mathrm{d}f_3}{\mathrm{d}\eta}\right) = (\beta - \alpha)f_2 + 2\alpha f_1' f_2' + \alpha(\beta - \alpha)f_1^2 + \alpha f_1 f_1'^2. \tag{16}$$

$$f_3'(1) + hf_3(1) = \alpha[f_1(1)f_2'(1) + f_1'(1)f_2(1)]. \quad (17)$$

In equations (14) and (16) the right-hand sides have been simplified by using the equation (12) for f_1.

Continuous solutions of (12), (13) and (14), (15) are given by

$$f_1 = \left(\frac{1-\eta^2}{4}\right) + \frac{\ln \eta_0}{2\gamma} \quad (18)$$

and

$$f_2 = \left(\frac{\beta - 2\alpha}{64}\right)\left(1-\eta^4\right) - \left(\frac{\beta-\alpha}{16}\right)\left(1-\eta^2\right)$$
$$-\left(\frac{\beta-\alpha}{8}\right)\left(\frac{\ln\eta_0}{\gamma}\right)\left(1-\eta^2\right) - \frac{\beta\ln\eta_0}{16\gamma} - \frac{\beta\ln^2\eta_0}{4\gamma^2}. \quad (19)$$

Upon evaluating C_0 by using equations (7), (11), (18), (19), we find that

$$\theta_i = \left[\frac{-J}{2\gamma} + J^2\left(\frac{\beta}{16\gamma} + \frac{\beta\ln\eta_0}{4\gamma^2}\right) + O(J^3)\right]\ln(\eta/\eta_0), \quad (20)$$

and clearly by using equations (18) and (19) in equation (11), we have determined $\theta(\eta)$ to terms of order $O(J^3)$.

When thick insulators with low thermal conductivities surround conductors, the temperature of the conductor becomes uniform and approaches the value

$$\theta = \frac{J\ln\eta_0}{2\gamma}. \quad (21)$$

Equation (21) is valid only for small values of J. However, when thick insulators with low thermal conductivities surround conductors, the temperature in a conductor may be assumed uniform throughout for *any* value of J. Then equation (2) is solved subject to the condition given by equation (5) together with the condition that

$$\theta' = \frac{-J(1-\beta\theta)}{2\gamma(1-\alpha\theta)} \quad \text{at} \quad \eta = 1. \quad (22)$$

The final result for the conductor temperature is,

$$\theta = \frac{2\gamma + J\beta\ln\eta_0 - [(2\gamma + J\beta\ln\eta_0)^2 - 8\alpha\gamma\ln\eta_0]^{\frac{1}{2}}}{4\alpha\gamma\ln\eta_0} \quad (23)$$

which is valid for all values of J.

(ii) *High Joule heating*

When $J \gg 1$, it is convenient to let $\epsilon = J^{-1}$. Thus, we now consider the singular perturbation problem of finding the asymptotic expansion as $\epsilon \to 0$ of the continuous solution of

$$\epsilon\frac{d}{d\eta}\left[\eta(1-\alpha\theta)\frac{d\theta}{d\eta}\right] + \eta(1-\beta\theta) = 0, \qquad 0 \leqslant \eta \leqslant 1, \quad (24)$$

which satisfies equation (9).

In the usual way we first assume a regular perturbation series of the form

$$\theta = \theta_0(\eta) + \epsilon\theta_1(\eta) + \epsilon^2\theta_2(\eta) + \ldots . \quad (25)$$

Substituting equation (25) into equation (24) and equating powers of ϵ, we immediately find that

$$\theta_0(\eta) = 1/\beta, \quad (26)$$

$$\theta_n(\eta) \equiv 0 \qquad \text{for} \quad n \geqslant 1. \quad (27)$$

Therefore, in the region $0 \leqslant \eta < 1$ except for a thermal boundary layer near $\eta = 1$, we find that to all powers of ϵ, $\theta(\eta) \sim 1/\beta$ as $\epsilon \to 0$. As we expect from the singular nature of the problem, the expansion equation (25) does not satisfy the boundary condition equation (9), and thus, we must find an expansion which is valid in the boundary layer near $\eta = 1$ and which matches with equation (26).

By standard techniques (see [4], for example) we find that the thermal boundary layer has a thickness of order $\epsilon^{\frac{1}{2}}$. Then, we define a new length $\tilde{\eta}$ given by

$$\tilde{\eta} = \frac{\eta - 1}{\epsilon^{\frac{1}{2}}} < 0, \quad (28)$$

and we assume the following expansion to be valid in the boundary layer:

$$\theta(\eta) = g_0(\tilde{\eta}) + \epsilon^{\frac{1}{2}}g_1(\tilde{\eta}) + \epsilon g_2(\tilde{\eta}) + \ldots . \quad (29)$$

Noting equation (26) and then substituting equation (29) into equations (9) and (24) and equating like powers of ϵ, we obtain

$$\frac{d}{d\tilde{\eta}}\left[(1-\alpha g_0)\frac{dg_0}{d\tilde{\eta}}\right] + (1-\beta g_0) = 0, \qquad -\infty \leqslant \tilde{\eta} \leqslant 0, \quad (30)$$

$$g_0'(0) = 0, \quad (31)$$

$$g_0(\tilde{\eta}) \to 1/\beta \qquad \text{as} \quad \tilde{\eta} \to -\infty, \quad (32)$$

$$\frac{d}{d\tilde{\eta}}\left[(1-\alpha g_0)\frac{dg_1}{d\tilde{\eta}} - \alpha\frac{dg_0}{d\tilde{\eta}}g_1\right] - \beta g_1 = \frac{1}{\eta}(g_0' - \alpha g_0 g_0'), \quad (33)$$

$$g_1'(0) = B, \quad (34)$$

$$g_1(\tilde{\eta}) \to 0 \qquad \text{as} \quad \tilde{\eta} \to -\infty, \quad (35)$$

$$\frac{d}{d\tilde{\eta}}\left[(1-\alpha g_0)\frac{dg_2}{d\tilde{\eta}} - \alpha\frac{dg_0}{d\tilde{\eta}}g_2\right] - \beta g_2 = \alpha(g_1g_1'' + g_1'^2)$$
$$-\frac{1}{\eta}(g_1' - \alpha g_0 g_1' - \alpha g_1 g_0'), \quad (36)$$

$$g_2'(0) = 0, \quad (37)$$

$$g_2(\tilde{\eta}) \to 0 \qquad \text{as} \quad \tilde{\eta} \to -\infty. \quad (38)$$

The boundary conditions equations (32), (35) and (38) express the proper matching conditions so that the boundary layer expansion matches with $1/\beta$ as ϵ approaches zero.

Clearly, a solution of equations (30)–(32) is given by

$$g_0(\tilde{\eta}) = 1/\beta. \tag{39}$$

By rigorous, but rather involved, phase-plane arguments it is possible to show [5] that there does exist another solution of equations (30)–(32). However, this other solution would imply a temperature at $\eta = 1$ which is greater than $1/\beta$; therefore this solution is rejected upon the grounds that such a temperature profile is impossible for the solution of the steady state conduction equations, (1)–(5).

Now we shall find $g_1(\tilde{\eta})$. Substituting equation (39) into equations (33) and (34) and solving for $g_1(\tilde{\eta})$ yields

$$g_1(\tilde{\eta}) = \frac{\gamma C_0(\beta - \alpha)^{\frac{1}{2}}}{\beta(1 + \alpha C_0 \ln \eta_0)} \exp\left[\beta\tilde{\eta}/(\beta - \alpha)^{\frac{1}{2}}\right]. \tag{40}$$

Therefore, in the boundary layer starting at $\eta = 1$ and penetrating to a distance of order $\epsilon^{\frac{1}{2}}$ the asymptotic expansion of the solution as $\epsilon \to 0$ is given by

$$\theta(\eta) = \frac{1}{\beta} - \epsilon^{\frac{1}{2}} \frac{\gamma C_0(\beta - \alpha)^{\frac{1}{2}}}{\beta(1 + \alpha C_0 \ln \eta_0)} \exp\left[\beta(\eta - 1)/(\beta - \alpha)^{\frac{1}{2}}\epsilon^{\frac{1}{2}} + O(\epsilon)\right] \tag{41}$$

Upon comparing equation (41) with equation (39) it can be seen that equation (41) is uniformly valid on the whole interval $0 \leqslant \eta \leqslant 1$.

It is now possible to determine the value of C_0 from equations (7) and (41); the result is

$$C_0 = \frac{-1}{\beta \ln \eta_0}\left[1 + \epsilon^{\frac{1}{2}} \frac{\gamma}{(\beta - \alpha)^{\frac{1}{2}}} + O(\epsilon)\right]. \tag{42}$$

Thus, the asymptotic expressions of the temperature profiles for the conductor and for the insulator are respectively

$$\theta(\eta) = \frac{1}{\beta} - \frac{1}{J^{\frac{1}{2}}} \frac{\gamma}{\beta(\beta - \alpha)^{\frac{1}{2}} \ln \eta_0} \exp\left\{\left[\frac{\beta J^{\frac{1}{2}}}{(\beta - \alpha)^{\frac{1}{2}}}\right](\eta - 1)\right\} + O\left(\frac{1}{J}\right), \quad 0 \leqslant \eta \leqslant 1, \tag{43}$$

and

$$\theta_I(\eta) = \frac{-1}{\beta \ln \eta_0}\left[1 + \frac{1}{J^{\frac{1}{2}}} \frac{\gamma}{(\beta - \alpha)^{\frac{1}{2}} \ln \eta_0} + O\left(\frac{1}{J}\right)\right] \ln\left(\frac{\eta}{\eta_0}\right), \quad 1 \leqslant \eta \leqslant \eta_0. \tag{44}$$

As expected, equations (23) and (43) both yield $\theta(\eta) = 1/\beta$ when J is large and γ is small.

REFERENCES

1. K. E. GREW and T. L. IBBS, *Thermal Diffusion in Gases*, p. 39. Cambridge University Press, Cambridge, England (1952).
2. D. S. REMER and F. H. SHAIR, Use of a tungsten filament lamp as a Pirani gauge for continuous gas analysis, *Rev. Scient. Instrum*, **40**, 968 (1969).
3. R. B. BIRD, W. E. STEWART and E. N. LIGHTFOOT, *Transport Phenomena*, p. 272. John Wiley, New York, (1960).
4. J. D. COLE, *Perturbation Methods in Applied Mathematics*. Blaisdell, Massachusetts (1968).
5. D. S. COHEN, Multiplicity of solutions of nonlinear boundary value problems. To be published.

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FUEL DROPLETS EVAPORATION IN A MOVING GASEOUS MEDIUM

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NOMENCLATURE

- A, area;
- C_D, drag coefficient;
- c_p, specific heat;
- D, diffusivity;
- h, heat-transfer coefficient;
- k, thermal conductivity;
- k_m, mass-transfer coefficient;
- M, mass of droplet;
- N_u, Nusselt number;
- P, pressure;
- Pr, Prandtl number;
- P_s, duct-static pressure;
- Q_t, total heat transfer to vapor film;
- Q_l, heat transfer to liquid droplet;
- Q_{sh}, heat carried with diffusing vapor in form of superheat;
- Q_λ, heat to vaporize diffusing vapor;
- Q_D, equals $Q_l + Q_\lambda$;
- R, gas constant;

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